BINOMIAL THEOREM

1. BINOMIAL EXPRESSION :

Any algebraic expression which contains two dissimilar terms is called binomial expression.

For example : x − y, xy + $\frac{1}{x}, \frac{1}{z} - 1, \frac{1}{(x-y)^{1/3}} + 3$ etc.

2. BINOMIAL THEOREM :

The formula by which any positive integral power of a binomial expression can be expanded in the form of a series is known as **BINOMIAL THEOREM.**

If x, y $\in$ R and n$\in$N, then : $(x + y)^n = {}^nC_0x^n + {}^nC_1x^{n-1}y + {}^nC_2x^{n-2}y^2 + \ldots + {}^nC_rx^{n-r}y^r + \ldots + {}^nC_ny^n = \sum_{r=0}^{n} {}^nC_r x^{n-r}y^r$

This theorem can be proved by induction.

Observations :

(a) The number of terms in the expansion is (n+1) i.e. one more than the index.

(b) The sum of the indices of x & y in each term is n.

(c) The binomial coefficients of the terms (${}^nC_0, {}^nC_1 \ldots$) equidistant from the beginning and the end are equal. i.e. ${}^nC_r = {}^nC_{r-1}$

(d) Symbol ${}^nC_r$ can also be denoted by $\binom{n}{r}$, C(n, r) or A_r^n .

Some important expansions :

(i) $(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots + {}^nC_nx^n$.

(ii) $(1 - x)^n = {}^nC_0 - {}^nC_1x + {}^nC_2x^2 + \ldots + (-1)^n . {}^nC_nx^n$.

Note : The coefficient of x^r in $(1+x)^n = {}^nC_r$ & that in $(1-x)^n = (-1)^r . {}^nC_r$

Illustration 1 : Expand : $(y + 2)^6$.

Solution : ${}^6C_0y^6 + {}^6C_1y^5.2 + {}^6C_2y^4.2^2 + {}^6C_3y^3.2^3 + {}^6C_4y^2.\ 2^4 + {}^6C_5y^1 . 2^5 + {}^6C_6 . 2^6$.

$= y^6 + 12y^5 + 60y^4 + 160y^3 + 240y^2 + 192y + 64$.

Illustration 2 : Write first 4 terms of $\left(1 - \frac{2y^2}{5}\right)^7$

Solution : ${}^7C_0, {}^7C_1\left(-\frac{2y^2}{5}\right), {}^7C_2\left(-\frac{2y^2}{5}\right)^2, {}^7C_3\left(-\frac{2y^2}{5}\right)^3$

Illustration 3 : The value of $\frac{\left(18^3 + 7^3 + 3.18.7.25\right)}{3^6 + 6.243.2 + 15.81.4 + 20.27.8 + 15.9.16 + 6.3.32 + 64}$ is -

(A) 1 (B) 2 (C) 3 (D) 4

Solution : The numerator is of the form $a^3 + b^3 + 3ab(a + b) = (a + b)^3$

Where, a = 18 and b = 7 $\therefore$ $N^r = (18 + 7)^3 = (25)^3$

Denominator can be written as

$$3^6 + {}^6C_1.3^5.2^1 + {}^6C_2.3^4.2^2 + {}^6C_3.3^3.2^3 + {}^6C_4\,3^2.2^4 + {}^6C_5\,3.2^5 + {}^6C_6\,2^6 = (3+2)^6 = 5^6 = (25)^3$$

$\therefore \frac{Nr}{Dr} = \frac{(25)^3}{(25)^3} = 1$ **Ans.**

Illustration 4 : If in the expansion of $(1 + x)^m (1 - x)^n$, the coefficients of x and x^2 are 3 and – 6 respectively then m is - **[JEE 99]**

(A) 6 (B) 9 (C) 12 (D) 24

Solution : $(1 + x)^m (1 - x)^n = \left[1 + mx + \frac{(m)(m-1).x^2}{2} + ...\right]\left[1 - nx + \frac{n(n-1)}{2}x^2 + ...\right]$

Coefficient of $x = m - n = 3$ ....(i)

Coefficient of $x^2 = -mn + \frac{n(n+1)}{2} + \frac{m(m-1)}{2} = -6$ ....(ii)

Solving (i) and (ii), we get

$m = 12$ and $n = 9$.

Do yourself - 1 :

(i) Expand $\left(3x^2 - \frac{x}{2}\right)^5$ **(ii)** Expand $(y + x)^n$

Pascal's triangle : A triangular arrangement of numbers as shown. The numbers give the coefficients for the expansion of $(x + y)^n$. The first row is for n = 0, the second for n = 1, etc. Each row has 1 as its first and last number. Other numbers are generated by adding the two numbers immediately to the left and right in the row above.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
etc.

3. IMPORTANT TERMS IN THE BINOMIAL EXPANSION :

(a) General term: The general term or the $(r+1)^{th}$ term in the expansion of $(x + y)^n$ is given by

$T_{r+1} = {}^nC_r\, x^{n-r}\, y^r$

Illustration 5 : Find : (a) The coefficient of x^7 in the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$

(b) The coefficient of x^{-7} in the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$

Also, find the relation between a and b, so that these coefficients are equal.

Solution : (a) In the expansion of $\left(ax^2 + \frac{1}{bx}\right)^{11}$, the general term is :

$$T_{r+1} = {}^{11}C_r(ax^2)^{11-r}\left(\frac{1}{bx}\right)^r = {}^{11}C_r.\frac{a^{11-r}}{b^r}.x^{22-3r}$$

putting $22 - 3r = 7$

$\therefore$ $3r = 15 \Rightarrow r = 5$

$\therefore$ $T_6 = {}^{11}C_5\frac{a^6}{b^5}.x^7$

Hence the coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ is ${}^{11}C_5a^6b^{-5}$. **Ans.**

Note that binomial coefficient of sixth term is ${}^{11}C_5$.

(b) In the expansion of $\left(ax - \frac{1}{bx^2}\right)^{11}$, general term is :

$$T_{r+1} = {}^{11}C_r (ax)^{11-r}\left(\frac{-1}{bx^2}\right)^r = (-1)^r\, {}^{11}C_r \frac{a^{11-r}}{b^r}.x^{11-3r}$$

putting $11 - 3r = -7$

$\therefore \quad 3r = 18 \quad \Rightarrow \quad r = 6$

$\therefore \quad T_7 = (-1)^6.\ {}^{11}C_6 \frac{a^5}{b^6}.x^{-7}$

Hence the coefficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$ is ${}^{11}C_6 a^5 b^{-6}$. **Ans.**

Also given :

Coefficient of x^7 in $\left(ax^2 + \frac{1}{bx}\right)^{11}$ = coefficient of x^{-7} in $\left(ax - \frac{1}{bx^2}\right)^{11}$

$\Rightarrow \quad {}^{11}C_5 a^6 b^{-5} = {}^{11}C_6 a^5 b^{-6}$

$\Rightarrow \quad ab = 1 \qquad (\because\ {}^{11}C_5 = {}^{11}C_6)$

which is the required relation between a and b. **Ans.**

Illustration 6 : Find the number of rational terms in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$.

Solution : The general term in the expansion of $(9^{1/4} + 8^{1/6})^{1000}$ is

$$T_{r+1} = {}^{1000}C_r \left(9^{\frac{1}{4}}\right)^{1000-r}\left(8^{\frac{1}{6}}\right)^r = {}^{1000}C_r\ 3^{\frac{1000-r}{2}}\, 2^{\frac{r}{2}}$$

The above term will be rational if exponents of 3 and 2 are integers

It means $\frac{1000-r}{2}$ and $\frac{r}{2}$ must be integers

The possible set of values of r is {0, 2, 4, ..., 1000}

Hence, number of rational terms is 501 **Ans.**

(b) Middle term :

The middle term(s) in the expansion of $(x + y)^n$ is (are) :

(i) If n is even, there is only one middle term which is given by $T_{(n+2)/2} = {}^nC_{n/2}.\ x^{n/2}.\ y^{n/2}$

(ii) If n is odd, there are two middle terms which are $T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$

Important Note :

Middle term has greatest binomial coefficient and if there are 2 middle terms their coefficients will be equal.

$\Rightarrow \quad {}^nC_r$ will be maximum

- When $r = \frac{n}{2}$ if n is even
- When $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$ if n is odd

$\Rightarrow$ The term containing greatest binomial coefficient will be middle term in the expansion of $(1 + x)^n$

Illustration 7 : Find the middle term in the expansion of $\left(3x-\frac{x^3}{6}\right)^9$

Solution : The number of terms in the expansion of $\left(3x-\frac{x^3}{6}\right)^9$ is 10 (even). So there are two middle terms.

i.e. $\left(\frac{9+1}{2}\right)^{th}$ and $\left(\frac{9+3}{2}\right)^{th}$ are two middle terms. They are given by T_5 and T_6

$$\therefore \quad T_5 = T_{4+1} = {}^9C_4(3x)^5\left(-\frac{x^3}{6}\right)^4 = {}^9C_4 3^5 x^5 . \frac{x^{12}}{6^4} = \frac{9.8.7.6}{1.2.3.4}.\frac{3^5}{2^4.3^4}x^{17} = \frac{189}{8}x^{17}$$

and $$T_6 = T_{5+1} = {}^9C_5(3x)^4\left(-\frac{x^3}{6}\right)^5 = -{}^9C_4 3^4.x^4.\frac{x^{15}}{6^5} = \frac{-9.8.7.6}{1.2.3.4}.\frac{3^4}{2^5.3^5}x^{19} = -\frac{21}{16}x^{19}$$ **Ans.**

(c) Term independent of x :

Term independent of x does not contain x ; Hence find the value of r for which the exponent of x is zero.

Illustration 8 : The term independent of x in $\left[\sqrt{\frac{x}{3}}+\sqrt{\left(\frac{3}{2x^2}\right)}\right]^{10}$ is -

(A) 1 (B) $\frac{5}{12}$ (C) ${}^{10}C_1$ (D) none of these

Solution : General term in the expansion is

$${}^{10}C_r\left(\frac{x}{3}\right)^{\frac{r}{2}}\left(\frac{3}{2x^2}\right)^{\frac{10-r}{2}} = {}^{10}C_r x^{\frac{3r}{2}-10}.\frac{3^{5-r}}{2^{\frac{10-r}{2}}}$$ For constant term, $\frac{3r}{2}=10 \Rightarrow r=\frac{20}{3}$

which is not an integer. Therefore, there will be no constant term. **Ans. (D)**

Do yourself - 2 :

(i) Find the 7th term of $\left(3x^2-\frac{1}{3}\right)^{10}$

(ii) Find the term independent of x in the expansion : $\left(2x^2-\frac{3}{x^3}\right)^{25}$

(iii) Find the middle term in the expansion of : (a) $\left(\frac{2x}{3}-\frac{3}{2x}\right)^6$ (b) $\left(2x^2-\frac{1}{x}\right)^7$

(d) Numerically greatest term :

Let numerically greatest term in the expansion of $(a + b)^n$ be T_{r+1}.

$$\Rightarrow \begin{cases} |T_{r+1}| \ge |T_r| \\ |T_{r+1}| \ge |T_{r+2}| \end{cases} \text{ where } T_{r+1} = {}^nC_r a^{n-r} b^r$$

Solving above inequalities we get $\frac{n+1}{1+\left|\frac{a}{b}\right|}-1 \le r \le \frac{n+1}{1+\left|\frac{a}{b}\right|}$

Case I : When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is an integer equal to m, then T_m and T_{m+1} will be numerically greatest term.

Case II : When $\frac{n+1}{1+\left|\frac{a}{b}\right|}$ is not an integer and its integral part is m, then T_{m+1} will be the numerically greatest term.

Illustration 9 : Find numerically greatest term in the expansion of $(3-5x)^{11}$ when $x=\frac{1}{5}$

Solution : Using $\frac{n+1}{1+\left|\frac{a}{b}\right|}-1\le r\le\frac{n+1}{1+\left|\frac{a}{b}\right|}$

$$\frac{11+1}{1+\left|\frac{3}{-5x}\right|}-1\le r\le\frac{11+1}{1+\left|\frac{3}{-5x}\right|}$$

solving we get $2\le r\le 3$

$\therefore\ r=2,3$

so, the greatest terms are T_{2+1} and T_{3+1}.

$\therefore$ Greatest term (when r = 2)

$T_3={}^{11}C_2.3^9(-5x)^2=55.3^9=T_4$

From above we say that the value of both greatest terms are equal. **Ans.**

Illustration 10 : Given T_3 in the expansion of $(1-3x)^6$ has maximum numerical value. Find the range of 'x'.

Solution : Using $\frac{n+1}{1+\left|\frac{a}{b}\right|}-1\le r\le\frac{n+1}{1+\left|\frac{a}{b}\right|}$

$$\frac{6+1}{1+\left|\frac{1}{-3x}\right|}-1\le 2\le\frac{7}{1+\left|\frac{1}{-3x}\right|}$$

Let $|x|=t$

$$\frac{21t}{3t+1}-1\le 2\le\frac{21t}{3t+1}$$

$$\left\{\begin{array}{l}\frac{21t}{3t+1}\le 3\\[4pt] \frac{21t}{3t+1}\ge 2\end{array}\right.\Rightarrow\left\{\begin{array}{l}\frac{4t-1}{3t+1}\le 0\Rightarrow t\in\left[-\frac{1}{3},\frac{1}{4}\right]\\[4pt] \frac{15t-2}{3t+1}\ge 0\Rightarrow t\in\left(-\infty,-\frac{1}{3}\right]\cup\left[\frac{2}{15},\infty\right)\end{array}\right.$$

Common solution $t\in\left[\frac{2}{15},\frac{1}{4}\right]\Rightarrow x\in\left[-\frac{1}{4},-\frac{2}{15}\right]\cup\left[\frac{2}{15},\frac{1}{4}\right]$

Do yourself -3 :

(i) Find the numerically greatest term in the expansion of $(3-2x)^9$, when $x=1$.

(ii) In the expansion of $\left(\frac{1}{2}+\frac{2x}{3}\right)^n$ when $x=-\frac{1}{2}$, it is known that 3^{rd} term is the greatest term. Find the possible integral values of n.

4. PROPERTIES OF BINOMIAL COEFFICIENTS :

$$(1+x)^n=C_0+C_1x+C_2x^2+C_3x^3+\ldots\ldots..+C_nx^n=\sum_{r=0}^{n}{}^nC_rr^r\ ;\ n\in N \quad \ldots.(i)$$

where $C_0,C_1,C_2,\ldots\ldots\ldots..C_n$ are called combinatorial (binomial) coefficients.

(a) The sum of all the binomial coefficients is 2^n.

Put x = 1, in (i) we get

$$C_0+C_1+C_2+\ldots\ldots\ldots..+C_n=2^n\Rightarrow\sum_{r=0}^{n}{}^nC_r=0 \quad \ldots.(ii)$$

(b) Put x=–1 in (i) we get

$$C_0-C_1+C_2-C_3\ldots\ldots\ldots.+C_n=0\Rightarrow\sum_{r=0}^{n}(-1)^r\,{}^nC_r=0 \quad \ldots(iii)$$

(c) The sum of the binomial coefficients at odd position is equal to the sum of the binomial coefficients at even position and each is equal to 2^{n-1}.

From (ii) & (iii), $C_0 + C_2 + C_4 + \ldots\ldots\ldots = C_1 + C_3 + C_5 + \ldots\ldots = 2^{n-1}$

(d) ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$

(e) $\dfrac{{}^nC_r}{{}^nC_{r-1}} = \dfrac{n-r+1}{r}$

(f) ${}^nC_r = \dfrac{n}{r}\,{}^{n-1}C_{r-1} = \dfrac{n}{r}.\dfrac{n-1}{r-1}\,{}^{n-2}C_{r-2} = \ldots\ldots = \dfrac{n(n-1)(n-2)\ldots\ldots(n-r+1)}{r(r-1)(r-2)\ldots\ldots\ldots 1}$

(g) ${}^nC_r = \dfrac{r+1}{n+1}.\,{}^{n+1}C_{r+1}$

Illustration 11 : Prove that : ${}^{25}C_{10} + {}^{24}C_{10} + \ldots\ldots + {}^{10}C_{10} = {}^{26}C_{11}$

Solution :

LHS $= {}^{10}C_{10} + {}^{11}C_{10} + {}^{12}C_{10} + \ldots\ldots\ldots\ldots + {}^{25}C_{10}$

$\Rightarrow {}^{11}C_{11} + {}^{11}C_{10} + {}^{12}C_{10} + \ldots\ldots + {}^{25}C_{10}$

$\Rightarrow {}^{12}C_{11} + {}^{12}C_{10} + \ldots\ldots + {}^{25}C_{10}$

$\Rightarrow {}^{13}C_{11} + {}^{13}C_{10} + \ldots\ldots\ldots {}^{25}C_{10}$

and so on. $\therefore$ LHS $= {}^{26}C_{11}$

Aliter :

LHS = coefficient of x^{10} in $\{(1+x)^{10} + (1+x)^{11} + \ldots\ldots\ldots\ldots (1+x)^{25}\}$

$\Rightarrow$ coefficient of x^{10} in $\left[(1+x)^{10}\dfrac{\{1+x\}^{16}-1}{1+x-1}\right]$

$\Rightarrow$ coefficient of x^{10} in $\dfrac{\left[(1+x)^{26}-(1+x)^{10}\right]}{x}$

$\Rightarrow$ coefficient of x^{11} in $\left[(1+x)^{26}-(1+x)^{10}\right] = {}^{26}C_{11} - 0 = {}^{26}C_{11}$

Illustration 12 : Prove that :

(i) $C_1 + 2C_2 + 3C_3 + \ldots\ldots + nC_n = n\,.\,2^{n-1}$

(ii) $C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \ldots\ldots\ldots + \dfrac{C_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$

Solution :

(i) L.H.S. $= \sum\limits_{r=1}^{n} r.\,{}^nC_r = \sum\limits_{r=1}^{n} r.\dfrac{n}{r}.\,{}^{n-1}C_{r-1}$

$= n\sum\limits_{r=1}^{n} {}^{n-1}C_{r-1} = n.\left[{}^{n-1}C_0 + {}^{n-1}C_1 + \ldots\ldots + {}^{n-1}C_{n-1}\right]$

$= n\,.\,2^{n-1}$

Aliter : (Using method of differentiation)

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots + {}^nC_nx^n$ ....(A)

Differentiating (A), we get

$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots\ldots + n.C_nx^{n-1}$.

Put $x = 1$,

$C_1 + 2C_2 + 3C_3 + \ldots\ldots\ldots + n.C_n = n.2^{n-1}$

(ii) L.H.S. $= \sum\limits_{r=0}^{n}\dfrac{C_r}{r+1} = \dfrac{1}{n+1}\sum\limits_{r=0}^{n}\dfrac{n+1}{r+1}\,{}^nC_r$

$= \dfrac{1}{n+1}\sum\limits_{r=0}^{n}{}^{n+1}C_{r+1} = \dfrac{1}{n+1}\left[{}^{n+1}C_1 + {}^{n+1}C_2 + \ldots\ldots + {}^{n+1}C_{n+1}\right] = \dfrac{1}{n+1}\left[2^{n+1}-1\right]$

Aliter : (Using method of integration)

Integrating (A), we get

$$\frac{(1+x)^{n+1}}{n+1}+C=C_0x+\frac{C_1x^2}{2}+\frac{C_2x^3}{3}+\dots\dots+\frac{C_nx^{n+1}}{n+1}\qquad\text{(where C is a constant)}$$

Put x = 0, we get, $C=-\frac{1}{n+1}$

$$\therefore\quad \frac{(1+x)^{n+1}-1}{n+1}=C_0x+\frac{C_1x^2}{2}+\frac{C_2x^3}{3}+\dots\dots+\frac{C_nx^{n+1}}{n+1}$$

Put x = 1, we get

$$C_0+\frac{C_1}{2}+\frac{C_2}{3}+\dots\dots\frac{C_n}{n+1}=\frac{2^{n+1}-1}{n+1}$$

Put x = –1, we get

$$C_0-\frac{C_1}{2}+\frac{C_2}{3}-\dots\dots=\frac{1}{n+1}$$

Illustration 13 : If $(1+x)^n=\sum_{r=0}^{n}{}^nC_rx^r$, then prove that $C_1^2+2.C_2^2+3.C_3^2+\dots\dots+n.C_n^2=\frac{(2n-1)!}{((n-1)!)^2}$

Solution : $(1+x)^n=C_0+C_1x+C_2x^2+C_2x^3+\dots\dots+C_n\,x^n$ ………(i)

Differentiating both the sides, w.r.t. x, we get

$n(1+x)^{n-1}=C_1+2C_2x+3C_2x^2+\dots\dots+n.C_nx^{n-1}$ ………(ii)

also, we have

$(x+1)^n=C_0x^n+C_1x^{n-1}+C_2x^{n-2}+\dots\dots+C_n$ ………(iii)

Multiplying (ii) & (iii), we get

$(C_1+2C_2x+3C_3x^2+\dots\dots+C_nx^{n-1})(C_0x^n+C_1x^{n-1}+C_2x^{n-2}+\dots\dots+C_n)=n(1+x)^{2n-1}$

Equating the coefficients of x^{n-1}, we get

$$C_1^2+2C_2^2+3C_3^2+\dots\dots+n.C_n^2=n.{}^{2n-1}C_{n-1}=\frac{(2n-1)!}{((n-1)!)^2}\qquad\textbf{Ans.}$$

Illustration 14 : Prove that : $C_0-3C_1+5C_2-\dots\dots(-1)^n(2n+1)C_n=0$

Solution : $T_r=(-1)^r(2r+1)^nC_r=2(-1)^rr\,.\,{}^nC_r+(-1)^r\,{}^nC_r$

$$\Sigma T_r=2\sum_{r=1}^{n}(-1)^r.r.\frac{n}{r}.{}^{n-1}C_{r-1}+\sum_{r=0}^{n}(-1)^r\ {}^nC_r=2\sum_{r=1}^{n}(-1)^r.{}^{n-1}C_{r-1}+\sum_{r=0}^{n}(-1)^r.{}^nC_r$$

$$=2\left[{}^{n-1}C_0-{}^{n-1}C_1+\dots..\right]+\left[{}^nC_0-{}^nC_1+\dots\dots\right]=0$$

Illustration 15 : Prove that $({}^{2n}C_0)^2-({}^{2n}C_1)^2+({}^{2n}C_2)^2-\dots.+(-1)^n({}^{2n}C_{2n})^2=(-1)^n.\ {}^{2n}C_n$

Solution : $(1-x)^{2n}={}^{2n}C_0-{}^{2n}C_1x+{}^{2n}C_2x^2-\dots.+(-1)^n\,{}^{2n}C_{2n}x^{2n}$ …(i)

and $(x+1)^{2n}={}^{2n}C_0x^{2n}+{}^{2n}C_1x^{2n-1}+{}^{2n}C_2x^{2n-2}+\dots+{}^{2n}C_{2n}$ ….(ii)

Multiplying (i) and (ii), we get

$(x^2-1)^{2n}=({}^{2n}C_0-{}^{2n}C_1x+\dots.+(-1)^n\,{}^{2n}C_{2n}x^{2n})\ ({}^{2n}C_0x^{2n}+{}^{2n}C_1x^{2n-1}+\dots.+{}^{2n}C_{2n})$ …. (iii)

Now, coefficient of x^{2n} in R.H.S.

$=({}^{2n}C_0)^2-({}^{2n}C_1)^2+({}^{2n}C_2)^2-\dots...+(-1)^n({}^{2n}C_{2n})^2$

$\because$ General term in L.H.S., $T_{r+1}={}^{2n}C_r(x^2)^{2n-r}(-1)^r$

Putting 2(2n – r) = 2n

$\therefore$ r = n

$\therefore$ $T_{n+1}={}^{2n}C_nx^{2n}(-1)^n$

Hence coeffiecient of x^{2n} in L.H.S. = $(-1)^n.{}^{2n}C_n$

But (iii) is an identity, therefore coefficient of x^{2n} in R.H.S. = coefficient of x^{2n} in L.H.S.

$\Rightarrow$ $({}^{2n}C_0)^2-({}^{2n}C_1)^2+({}^{2n}C_2)^2-\dots.+(-1)^n({}^{2n}C_{2n})^2=(-1)^n.\ {}^{2n}C_n$

Illustration 16 : Prove that : ${}^nC_0.{}^{2n}C_n - {}^nC_1.{}^{2n-2}C_n + {}^nC_2.{}^{2n-4}C_n + \ldots = 2^n$

Solution : L.H.S. = Coefficient of x^n in $[{}^nC_0(1+x)^{2n} - {}^nC_1(1+x)^{2n-2} \ldots\ldots]$

= Coefficient of x^n in $[(1+x)^2 - 1]^n$

= Coefficient of x^n in $x^n(x+2)^n = 2^n$

Illustration 17 : If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$ then show that the sum of the products of the C_i's taken two at a time represented by : $\sum\sum_{0\le i<j\le n} C_iC_j$ is equal to $2^{2n-1} - \dfrac{2n!}{2.n!n!}$

Solution : Since $(C_0 + C_1 + C_2 + \ldots + C_{n-1} + C_n)^2$

$= C_0^2 + C_1^2 + C_2^2 + \ldots + C_{n-1}^2 + C_n^2 + 2(C_0C_1 + C_0C_2 + C_0C_3 + \ldots + C_0C_n + C_1C_2 + C_1C_3 + \ldots + C_1C_n + C_2C_3 + C_2C_4 + \ldots + C_2C_n + \ldots + C_{n-1}C_n)$

$(2^n)^2 = {}^{2n}C_n + 2\sum\sum_{0\le i<j\le n} C_iC_j$

Hence $\sum\sum_{0\le i<j\le n} C_iC_j = 2^{2n-1} - \dfrac{2n!}{2.n!n!}$ **Ans.**

Illustration 18 : If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + C_nx^n$ then prove that $\sum\sum_{0\le i<j\le n}(C_i + C_j)^2 = (n-1)\,{}^{2n}C_n + 2^{2n}$

Solution : L.H.S $\sum\sum_{0\le i<j\le n}(C_i + C_j)^2$

$= (C_0 + C_1)^2 + (C_0 + C_2)^2 + \ldots + (C_0 + C_n)^2 + (C_1 + C_2)^2 + (C_1 + C_3)^2 + \ldots + (C_1 + C_n)^2 + (C_2 + C_3)^2 + (C_2 + C_4)^2 + \ldots + (C_2 + C_n)^2 + \ldots + (C_{n-1} + C_n)^2$

$= n(C_0^2 + C_1^2 + C_2^2 + \ldots + C_n^2) + 2\sum\sum_{0\le i<j\le n} C_iC_j$

$= n.{}^{2n}C_n + 2.\left\{2^{2n-1} - \dfrac{2n!}{2.n!n!}\right\}$ {from Illustration 17}

$= n.{}^{2n}C_n + 2^{2n} - {}^{2n}C_n = (n-1).{}^{2n}C_n + 2^{2n}$ = R.H.S.

Do yourself - 4 :

(i) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots\ldots + \binom{n}{n} =$

(A) 2^{n-1} (B) ${}^{2n}C_n$ (C) 2^n (D) 2^{n+1}

(ii) If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$, $n \in N$. Prove that

(a) $3C_0 - 8C_1 + 13C_2 - 18C_3 + \ldots\ldots$ upto $(n+1)$ terms $= 0$, if $n \ge 2$.

(b) $2C_0 + 2^2\dfrac{C_1}{2} + 2^3\dfrac{C_2}{3} + 2^4\dfrac{C_3}{4} + \ldots\ldots + 2^{n+1}\dfrac{C_n}{n+1} = \dfrac{3^{n+1}-1}{n+1}$

(c) $C_0^2 + \dfrac{C_1^2}{2} + \dfrac{C_2^2}{3} + \ldots\ldots + \dfrac{C_n^2}{n+1} = \dfrac{(2n+1)!}{((n+1)!)^2}$

5. MULTINOMIAL THEOREM :

Using binomial theorem, we have $(x+a)^n = \sum_{r=0}^{n} {}^nC_r x^{n-r}a^r,\ n \in N$

$= \sum_{r=0}^{n}\dfrac{n!}{(n-r)!r!}x^{n-r}a^r = \sum_{r+s=n}\dfrac{n!}{r!s!}x^sa^r$, where $s + r = n$

This result can be generalized in the following form.

$$(x_1 + x_2 + \ldots\ldots + x_k)^n = \sum_{r_1+r_2+\ldots+r_k=n}\frac{n!}{r_1!r_2!\ldots r_k!}x_1^{r_1}x_2^{r_2}\ldots\ldots x_k^{r_k}$$

The general term in the above expansion $\dfrac{n!}{r_1!\,r_2!\,r_3!\ldots\ldots r_k!}.x_1^{r_1}x_2^{r_2}x_3^{r_3}\ldots\ldots x_k^{r_k}$

The number of terms in the above expansion is equal to the number of non-negative integral solution of the equation $r_1 + r_2 + \ldots\ldots + r_k = n$ because each solution of this equation gives a term in the above expansion. The number of such solutions is ${}^{n+k-1}C_{k-1}$

Particular cases :

(i) $(x + y + z)^n = \displaystyle\sum_{r+s+t=n} \frac{n!}{r!s!t!} x^r y^s z^t$

The above expansion has ${}^{n+3-1}C_{3-1} = {}^{n+2}C_2$ terms

(ii) $(x + y + z + u)^n = \displaystyle\sum_{p+q+r+s=n} \frac{n!}{p!q!r!s!} x^p y^q z^r u^s$

There are ${}^{n+4-1}C_{4-1} = {}^{n+3}C_3$ terms in the above expansion.

Illustration 19 : Find the coefficient of $x^2y^3z^4w$ in the expansion of $(x - y - z + w)^{10}$

Solution : $(x - y - z + w)^{10} = \displaystyle\sum_{p+q+r+s=10} \frac{n!}{p!q!r!s!}(x)^p(-y)^q(-z)^r(w)^s$

We want to get $x^2y^3z^4w$ this implies that $p = 2,\ q = 3,\ r = 4,\ s = 1$

$\therefore$ Coefficient of $x^2y^3z^4w$ is $\dfrac{10!}{2!\,3!\,4!\,1!}(-1)^3(-1)^4 = -12600$ **Ans.**

Illustration 20 : Find the total number of terms in the expansion of $(1 + x + y)^{10}$ and coefficient of x^2y^3.

Solution : Total number of terms $= {}^{10+3-1}C_{3-1} = {}^{12}C_2 = 66$

Coefficient of $x^2y^3 = \dfrac{10!}{2!\times 3!\times 5!} = 2520$ **Ans.**

Illustration 21 : Find the coefficient of x^5 in the expansion of $(2 - x + 3x^2)^6$.

Solution : The general term in the expansion of $(2 - x + 3x^2)^6 = \dfrac{6!}{r!s!t!}2^r(-x)^s(3x^2)^t$, where $r + s + t = 6$.

$= \dfrac{6!}{r!s!t!}2^r \times (-1)^s \times (3)^t \times x^{s+2t}$

For the coefficient of x^5, we must have $s + 2t = 5$.

But, $r + s + t = 6$,

$\therefore$ $s = 5 - 2t$ and $r = 1 + t$, where $0 \le r, s, t \le 6$.

Now $t = 0 \Rightarrow r = 1,\ s = 5.$

$t = 1 \Rightarrow r = 2,\ s = 3.$

$t = 2 \Rightarrow r = 3,\ s = 1.$

Thus, there are three terms containing x^5 and coefficient of x^5

$= \dfrac{6!}{1!\,5!\,0!}\times 2^1 \times (-1)^5 \times 3^0 + \dfrac{6!}{2!\,3!\,1!}\times 2^2 \times (-1)^3 \times 3^1 + \dfrac{6!}{3!\,1!\,2!}\times 2^3 \times (-1)^1 \times 3^2$

$= -12 - 720 - 4320 = -5052.$ **Ans.**

Illustration 22 : If $(1+x+x^2)^n = \sum_{r=0}^{2n} a_r x^r$, then prove that (a) $a_r = a_{2n-r}$ (b) $\sum_{r=0}^{n-1} a_r = \frac{1}{2}(3^n - a_n)$

Solution : (a) We have

$$\left(1+x+x^2\right)^n = \sum_{r=0}^{2n} a_r x^r \quad \text{....(A)}$$

Replace x by $\frac{1}{x}$

$$\therefore \quad \left(1+\frac{1}{x}+\frac{1}{x^2}\right)^n = \sum_{r=0}^{2n} a_r \left(\frac{1}{x}\right)^r$$

$$\Rightarrow \quad \left(x^2+x+1\right)^n = \sum_{r=0}^{2n} a_r x^{2n-r}$$

$$\Rightarrow \quad \sum_{r=0}^{2n} a_r x^r = \sum_{r=0}^{2n} a_r x^{2n-r} \quad \{\text{Using (A)}\}$$

Equating the coefficient of x^{2n-r} on both sides, we get

$a_{2n-r} = a_r$ for $0 \leq r \leq 2n$.

Hence $a_r = a_{2n-r}$.

(b) Putting x=1 in given series, then

$a_0 + a_1 + a_2 + \ldots\ldots\ldots + a_{2n} = (1+1+1)^n$

$a_0 + a_1 + a_2 + \ldots\ldots\ldots + a_{2n} = 3^n$ ....(1)

But $a_r = a_{2n-r}$ for $0 \leq r \leq 2n$

$\therefore$ series (1) reduces to

$2(a_0 + a_1 + a_2 + \ldots\ldots + a_{n-1}) + a_n = 3^n$.

$\therefore \quad a_0 + a_1 + a_2 + \ldots\ldots + a_{n-1} = \frac{1}{2}(3^n - a_n)$

Do yourself - 5 :

(i) Find the coefficient of x^2y^5 in the expansion of $(3 + 2x - y)^{10}$.

6. APPLICATION OF BINOMIAL THEOREM :

Illustration 23 : If $\left(6\sqrt{6}+14\right)^{2n+1} = [N] + F$ and $F = N - [N]$; where [.] denotes greatest integer function, then NF is equal to

(A) 20^{2n+1} (B) an even integer (C) odd integer (D) 40^{2n+1}

Solution : Since $\left(6\sqrt{6}+14\right)^{2n+1} = [N] + F$

Let us assume that $f = \left(6\sqrt{6}-14\right)^{2n+1}$; where $0 \leq f < 1$.

Now, $[N] + F - f = \left(6\sqrt{6}+14\right)^{2n+1} - \left(6\sqrt{6}-14\right)^{2n+1}$

$= 2\left[{}^{2n+1}C_1\left(6\sqrt{6}\right)^{2n}(14) + {}^{2n+1}C_3\left(6\sqrt{6}\right)^{2n-2}(14)^3 + \ldots\right]$

$\Rightarrow$ $[N] + F - f$ = even integer.

Now $0 < F < 1$ and $0 < f < 1$

so $-1 < F - f < 1$ and $F - f$ is an integer so it can only be zero

Thus $NF = \left(6\sqrt{6}+14\right)^{2n+1}\left(6\sqrt{6}-14\right)^{2n+1} = 20^{2n+1}$. **Ans. (A,B)**

Illustration 24 : Find the last three digits in 11^{50}.

Solution : Expansion of $(10 + 1)^{50} = {}^{50}C_0 10^{50} + {}^{50}C_1 10^{49} + \ldots.. + {}^{50}C_{48} 10^2 + {}^{50}C_{49} 10 + {}^{50}C_{50}$

$= \underbrace{{}^{50}C_0 10^{50} + {}^{50}C_1 10^{49} + \ldots\ldots + {}^{50}C_{47} 10^3}_{1000K} + 49 \quad 25 \quad 100 + 500 + 1$

$\Rightarrow \quad 1000\,K + 123001$

$\Rightarrow$ Last 3 digits are 001.

Illustration 25 : Prove that $2222^{5555} + 5555^{2222}$ is divisible by 7.

Solution : When 2222 is divided by 7 it leaves a remainder 3. So adding & subtracting 3^{5555}, we get :

$E = \underbrace{2222^{5555} - 3^{5555}}_{E_1} + \underbrace{3^{5555} + 5555^{2222}}_{E_2}$

For E_1 : Now since $2222 - 3 = 2219$ is divisible by 7, therefore E_1 is divisible by 7

($\because$ $x^n - a^n$ is divisible by $x - a$)

For E_2 : 5555 when devided by 7 leaves remainder 4. So adding and subtracting 4^{2222}, we get :

$E_2 = 3^{5555} + 4^{2222} + 5555^{2222} - 4^{2222}$

$= (243)^{1111} + (16)^{1111} + (5555)^{2222} - 4^{2222}$

Again $(243)^{1111} + 16^{1111}$ and $(5555)^{2222} - 4^{2222}$ are divisible by 7

($\because$ $x^n + a^n$ is divisible by $x + a$ when n is odd)

Hence $2222^{5555} + 5555^{2222}$ is divisible by 7.

Do yourself - 6 :

(i) Prove that $5^{25} - 3^{25}$ is divisible by 2.

(ii) Find the remainder when the number 9^{100} is divided by 8.

(iii) Find last three digits in 19^{100}.

(iv) Let $R = (8 + 3\sqrt{7})^{20}$ and [.] denotes greatest integer function, then prove that :

(a) [R] is odd (b) $R - [R] = 1 - \dfrac{1}{(8 + 3\sqrt{7})^{20}}$

(v) Find the digit at unit's place in the number $17^{1995} + 11^{1995} - 7^{1995}$.

7. BINOMIAL THEOREM FOR NEGATIVE OR FRACTIONAL INDICES :

If $n \in Q$, then $(1 + x)^n = 1 + nx + \dfrac{n(n-1)}{2!} x^2 + \dfrac{n(n-1)(n-2)}{3!} x^3 + \ldots\ldots. \infty$ provided $|x| < 1$.

Note :

(i) When the index n is a positive integer the number of terms in the expansion of $(1 + x)^n$ is finite i.e. (n+1) & the coefficient of successive terms are : ${}^nC_0, {}^nC_1, {}^nC_2, \ldots\ldots. {}^nC_n$

(ii) When the index is other than a positive integer such as negative integer or fraction, the number of terms in the expansion of $(1 + x)^n$ is infinite and the symbol ${}^nC_r$ cannot be used to denote the coefficient of the general term.

(iii) Following expansion should be remembered ($|x| < 1$).

(a) $(1 + x)^{-1} = 1 - x + x^2 - x^3 + x^4 - \ldots. \infty$

(b) $(1 - x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots. \infty$

(c) $(1 + x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \ldots. \infty$

(d) $(1 - x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots. \infty$

(e) $(1 + x)^{-3} = 1 - 3x + 6x^2 - 10x^3 + \ldots.. + \dfrac{(-1)^r (r+1)(r+2)}{2!} x^r + \ldots\ldots.$

(f) $(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + ... + \frac{(r+1)(r+2)}{2!}x^r + ...$

(iv) The expansions in ascending powers of x are only valid if x is 'small'. If x is large i.e. $|x|>1$ then we may find it convenient to expand in powers of $1/x$, which then will be small.

8. APPROXIMATIONS :

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 ....$$

If $x < 1$, the terms of the above expansion go on decreasing and if x be very small, a stage may be reached when we may neglect the terms containing higher powers of x in the expansion. Thus, if x be so small that its square and higher powers may be neglected then $(1+x)^n = 1 + nx$, approximately.

This is an approximate value of $(1+x)^n$

Illustration 26 : If x is so small such that its square and higher powers may be neglected then find the approximate value of $\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}}$

Solution :

$$\frac{(1-3x)^{1/2}+(1-x)^{5/3}}{(4+x)^{1/2}} = \frac{1-\frac{3}{2}x+1-\frac{5x}{3}}{2\left(1+\frac{x}{4}\right)^{1/2}} = \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1+\frac{x}{4}\right)^{-1/2} = \frac{1}{2}\left(2-\frac{19}{6}x\right)\left(1-\frac{x}{8}\right)$$

$$= \frac{1}{2}\left(2-\frac{x}{4}-\frac{19}{6}x\right) = 1-\frac{x}{8}-\frac{19}{12}x = 1 - \frac{41}{24}x$$ **Ans.**

Illustration 27 : The value of cube root of 1001 upto five decimal places is –

(A) 10.03333 (B) 10.00333 (C) 10.00033 (D) none of these

Solution :

$$(1001)^{1/3} = (1000+1)^{1/3} = 10\left(1+\frac{1}{1000}\right)^{1/3} = 10\left\{1+\frac{1}{3}.\frac{1}{1000}+\frac{1/3(1/3-1)}{2!}\frac{1}{1000^2}+.....\right\}$$

$= 10\{1 + 0.0003333 - 0.00000011 + ...\} = 10.00333$ **Ans. (B)**

Illustration 28 : The sum of $1 + \frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+....\infty$ is -

(A) $\sqrt{2}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\sqrt{3}$ (D) $2^{3/2}$

Solution : Comparing with $1 + nx + \frac{n(n-1)}{2!}x^2 + ...$

$nx = 1/4$ .... (i)

and $\frac{n(n-1)x^2}{2!} = \frac{1.3}{4.8}$

or $\frac{nx(nx-x)}{2!} = \frac{3}{32} \Rightarrow \frac{1}{4}\left(\frac{1}{4}-x\right) = \frac{3}{16}$ (by (i))

$\Rightarrow \left(\frac{1}{4}-x\right) = \frac{3}{4} \Rightarrow x = \frac{1}{4}-\frac{3}{4} = -\frac{1}{2}$ ....(ii)

putting the value of x in (i)

$n(-1/2) = 1/4 \Rightarrow n = -1/2$

$\therefore$ sum of series $= (1+x)^n = (1-1/2)^{-1/2} = (1/2)^{-1/2} = \sqrt{2}$ **Ans. (A)**

9. EXPONENTIAL SERIES :

(a) e is an irrational number lying between 2.7 & 2.8. Its value correct upto 10 places of decimal is 2.7182818284.

(b) Logarithms to the base 'e' are known as the Napierian system, so named after Napier, their inventor. They are also called **Natural Logarithm.**

(c) $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+.......\infty$; where x may be any real or complex number & $e = \lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

(d) $a^x = 1+\frac{x}{1!}\ell na+\frac{x^2}{2!}\ell n^2a+\frac{x^3}{3!}\ell n^3a+.......\infty$, where $a > 0$

(e) $e = 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+.......\infty$

10. LOGARITHMIC SERIES :

(a) $\ell n\,(1 + x) = x - \frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+.......\infty$, where $-1 < x \le 1$

(b) $\ell n\,(1 - x) = - x - \frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}+.......\infty$, where $-1 \le x < 1$

Remember : (i) $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+.......\infty = \ell n 2$ (ii) $e^{\ln x} = x$; for all $x > 0$

(iii) $\ell n2 = 0.693$ (iv) $\ell n10 = 2.303$

ANSWERS FOR DO YOURSELF

1. (i) ${}^5C_0x\,(3x^2)^5 + {}^5C_1(3x^2)^4\left(-\frac{x}{2}\right) + {}^5C_2(3x^2)^3\left(-\frac{x}{2}\right)^2 + {}^5C_3(3x^2)^2\left(-\frac{x}{2}\right)^3 + {}^5C_4(3x^2)^1\left(-\frac{x}{2}\right)^4 + {}^5C_5\left(-\frac{x}{2}\right)^5$

(ii) ${}^nC_0y^n + {}^nC_1y^{n-1}.x + {}^nC_2.y^{n-2}.x^2 + ...+{}^nC_n.x^n$

2 : (i) $\frac{70}{3}x^8$; (ii) $\frac{25!}{10!\ 5!}2^{15}3^{10}$; (iii) (a) -20; (b) $-560x^5$, $280x^2$

3. (i) 4^{th} & 5^{th} i.e. 489888 (ii) n = 4, 5, 6

4. (i) C

5. (i) -272160 or $-{}^{10}C_5\ {}^5C_2\ 108$

6. (ii) 1 (iii) 801 (v) 1

EXERCISE - 01 CHECK YOUR GRASP

SELECT THE CORRECT ALTERNATIVE (ONLY ONE CORRECT ANSWER)

1. If the coefficients of x^7 & x^8 in the expansion of $\left[2+\frac{x}{3}\right]^n$ are equal, then the value of n is -

(A) 15 (B) 45 (C) 55 (D) 56

2. The sum of the binomial coefficients of $\left[2x+\frac{1}{x}\right]^n$ is equal to 256. The constant term in the expansion is -

(A) 1120 (B) 2110 (C) 1210 (D) none

3. The sum of the co–efficients in the expansion of $(1 - 2x + 5x^2)^n$ is 'a' and the sum of the co–efficients in the expansion of $(1 + x)^{2n}$ is b. Then -

(A) $a = b$ (B) $a = b^2$ (C) $a^2 = b$ (D) $ab = 1$

4. Given that the term of the expansion $(x^{1/3} - x^{-1/2})^{15}$ which does not contain x is 5 m where $m \in N$, then m is equal to -

(A) 1100 (B) 1010 (C) 1001 (D) none

5. The expression $\frac{1}{\sqrt{4x+1}}\left[\left[\frac{1+\sqrt{4x+1}}{2}\right]^7-\left[\frac{1-\sqrt{4x+1}}{2}\right]^7\right]$ is a polynomial in x of degree -

(A) 7 (B) 5 (C) 4 (D) 3

6. In the binomial $(2^{1/3} + 3^{-1/3})^n$, if the ratio of the seventh term from the beginning of the expansion to the seventh term from its end is 1/6, then n is equal to -

(A) 6 (B) 9 (C) 12 (D) 15

7. The term independent of x in the product $(4+x+7x^2)\left(x-\frac{3}{x}\right)^{11}$ is -

(A) $7.{}^{11}C_6$ (B) $3^6.\ {}^{11}C_6$ (C) $3^5.\ {}^{11}C_5$ (D) $-12.\ 2^{11}$

8. If 'a' be the sum of the odd terms & 'b' be the sum of the even terms in the expansion of $(1+x)^n$, then $(1-x)^n$ is equal to -

(A) $a - b$ (B) $a + b$ (C) $b - a$ (D) none

9. The sum of the co–efficients of all the even powers of x in the expansion of $(2x^2 - 3x + 1)^{11}$ is -

(A) $2 . 6^{10}$ (B) $3 . 6^{10}$ (C) 6^{11} (D) none

10. The greatest terms of the expansion $(2x + 5y)^{13}$ when x = 10, y = 2 is -

(A) ${}^{13}C_5 . 20^8 . 10^5$ (B) ${}^{13}C_6 . 20^7 . 10^4$ (C) ${}^{13}C_4 . 20^9 . 10^4$ (D) none of these

11. Number of rational terms in the expansion of $\left(\sqrt{2}+\sqrt[4]{3}\right)^{100}$ is -

(A) 25 (B) 26 (C) 27 (D) 28

12. If $\binom{p}{q}=0$ for p < q, where p, q $\in$ W, then $\sum_{r=0}^{\infty}\binom{n}{2r}=$

(A) 2^n (B) 2^{n-1} (C) 2^{2n-1} (D) ${}^{2n}C_n$

13. $\binom{47}{4}+\sum_{j=1}^{5}\binom{52-j}{3}=\binom{x}{y}$, then $\frac{x}{y}=$

(A) 11 (B) 12 (C) 13 (D) 14

14. If $n \in N$ & n is even, then $\frac{1}{1.(n-1)!}+\frac{1}{3!(n-3)!}+\frac{1}{5!(n-5)!}+......+\frac{1}{(n-1)!1!}=$

(A) 2^n (B) $\frac{2^{n-1}}{n!}$ (C) $2^n n!$ (D) none of these

15. Let $R=(5\sqrt{5}+11)^{31}=I+f$, where I is an integer and f is the fractional part of R, then R f is equal to -

(A) 2^{31} (B) 3^{31} (C) 2^{62} (D) 1

16. The value of $\sum_{r=0}^{10}\binom{10}{r}\binom{15}{14-r}$ is equal to -

(A) ${}^{25}C_{12}$ (B) ${}^{25}C_{15}$ (C) ${}^{25}C_{10}$ (D) ${}^{25}C_{11}$

17. $\frac{C_0}{1}+\frac{C_1}{2}+\frac{C_2}{3}+......+\frac{C_{10}}{11}$ is equal to (here $C_r = {}^{10}C_r$)

(A) $\frac{2^{11}}{11}$ (B) $\frac{2^{11}-1}{11}$ (C) $\frac{3^{11}}{11}$ (D) $\frac{3^{11}-1}{11}$

18. If $a_n=\sum_{r=0}^{n}\frac{1}{{}^nC_r}$, then $\sum_{r=0}^{n}\frac{r}{{}^nC_r}$ equals - [JEE 98]

(A) $(n-1)a_n$ (B) $n\,a_n$ (C) $n\,a_n/2$ (D) none of these

19. The last two digits of the number 3^{400} are -

(A) 81 (B) 43 (C) 29 (D) 01

<u>SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)</u>

20. If the coefficients of three consecutive terms in the expansion of $(1 + x)^n$ are in the ratio of 1 : 7 : 42, then n is divisible by -

(A) 9 (B) 5 (C) 3 (D) 11

21. In the expansion of $\left(\sqrt[3]{4}+\frac{1}{\sqrt[4]{6}}\right)^{20}$ -

(A) the number of irrational terms = 19
(B) middle term is irrational
(C) the number of rational terms = 2
(D) 9^{th} term is rational

22. If $(1 + x + x^2 + x^3)^{100} = a_0 + a_1x + a_2x^2 + ... + a_{300}x^{300}$, then -

(A) $a_0 + a_1 + a_2 + a_3 +.......+ a_{300}$ is divisible by 1024
(B) $a_0 + a_2 + a_4 +.......+ a_{300} = a_1 + a_3 + ... + a_{299}$
(C) coefficients equidistant from beginning and end are equal
(D) $a_1 = 100$

23. The number $101^{100} - 1$ is divisible by -

(A) 100 (B) 1000 (C) 10000 (D) 100000

24. If $\left(9+\sqrt{80}\right)^n = I + f$ where I, n are integers and $0 < f < 1$, then -

(A) I is an odd integer
(B) I is an even integer
(C) $(I + f)(1 - f) = 1$
(D) $1 - f = \left(9-\sqrt{80}\right)^n$

25. In the expansion of $\left(x^{2/3}-\frac{1}{\sqrt{x}}\right)^{30}$, a term containing the power x^{13} -

(A) does not exist
(B) exists and the co-efficient is divisible by 29
(C) exists and the co-efficient is divisible by 63
(D) exists and the co-efficient is divisible by 65

26. The co-efficient of the middle term in the expansion of $(1+x)^{2n}$ is -

(A) $\dfrac{1.3.5.7......(2n-1)}{n!}\,2^n$

(B) ${}^{2n}C_n$

(C) $\dfrac{(n+1)(n+2)(n+3)....(2n-1)(2n)}{1.2.3.........(n-1)\,n}$

(D) $\dfrac{2.6.10.14......(4n-6)(4n-2)}{1.2.3.4.....(n-1).n}$

CHECK YOUR GRASP — ANSWER KEY — EXERCISE-1

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	C	A	A	C	D	B	B	A	B	C
Que.	11	12	13	14	15	16	17	18	19	20
Ans.	B	B	C	B	C	D	B	C	D	B,D
Que.	21	22	23	24	25	26				
Ans.	A,B,C,D	A,B,C,D	A,B,C	A,C,D	B,C,D	A,B,C,D				

EXERCISE - 02

BRAIN TEASERS

SELECT THE CORRECT ALTERNATIVES (ONE OR MORE THAN ONE CORRECT ANSWERS)

1. The coefficient of x^r $(0 \le r \le n-1)$ in the expression :
$(x+2)^{n-1} + (x+2)^{n-2} . (x+1) + (x+2)^{n-3} . (x+1) + \ldots\ldots + (x+1)^{n-1}$ is -

(A) ${}^nC_r(2^r - 1)$ (B) ${}^nC_r(2^{n-r} - 1)$ (C) ${}^nC_r(2^r + 1)$ (D) ${}^nC_r(2^{n-r} + 1)$

2. If $(1 + x + x\ )^{25} = a_0 + a_1x + a_2x + \ldots.. + a_{50} . x^{50}$ then $a_0 + a_2 + a_4 + \ldots.. + a_{50}$ is -

(A) even (B) odd & of the form 3n

(C) odd & of the form $(3n-1)$ (D) odd & of the form $(3n+1)$

3. The co-efficient of x^4 in the expansion of $(1 - x + 2x^2)^{12}$ is -

(A) ${}^{12}C_3$ (B) ${}^{13}C_3$ (C) ${}^{14}C_4$ (D) ${}^{12}C_3 + 3\ {}^{13}C_3 + {}^{14}C_4$

4. Let $(1 + x^2)^2 (1 + x)^n = A_0 + A_1x + A_2x^2 + \ldots\ldots$ If A_0, A_1, A_2 are in A.P. then the value of n is -

(A) 2 (B) 3 (C) 5 (D) 7

5. If $\sum\limits_{k=1}^{n-r} {}^{n-k}C_r = {}^xC_y$ then -

(A) $x = n + 1$; $y = r$ (B) $x = n$; $y = r + 1$

(C) $x = n$; $y = r$ (D) $x = n + 1$; $y = r + 1$

6. Co-efficient of α^t in the expansion of $(\alpha + p)^{m-1} + (\alpha + p)^{m-2} (\alpha + q) + (\alpha + p)^{m-3} (\alpha + q)^2 + \ldots\ldots (\alpha + q)^{m-1}$ where $\alpha \ne -q$ and $p \ne q$ is -

(A) $\dfrac{{}^mC_t\left(p^t - q^t\right)}{p-q}$ (B) $\dfrac{{}^mC_t\left(p^{m-t} - q^{m-t}\right)}{p-q}$ (C) $\dfrac{{}^mC_t\left(p^t + q^t\right)}{p-q}$ (D) $\dfrac{{}^mC_t\left(p^{m-t} + q^{m-t}\right)}{p-q}$

7. The co-efficient of x^{401} in the expansion of $(1 + x + x^2 + \ldots\ldots + x^9)^{-1}$, $(|x| < 1)$ is -

(A) 1 (B) -1 (C) 2 (D) -2

8. Number of terms free from radical sign in the expansion of $(1 + 3^{1/3} + 7^{1/7})^{10}$ is -

(A) 4 (B) 5 (C) 6 (D) 8

9. The value r for which $\binom{30}{r}\binom{15}{r} + \binom{30}{r-1}\binom{15}{1} + \ldots\ldots. + \binom{30}{0}\binom{15}{r}$ is maximum is/are -

(A) 21 (B) 22 (C) 23 (D) 24

10. If the 6^{th} term in the expansion of $\left(\dfrac{3}{2} + \dfrac{x}{3}\right)^n$ when $x = 3$ is numerically greatest then the possible integral value(s) of n can be -

(A) 11 (B) 12 (C) 13 (D) 14

11. In the expansion of $(1 + x)^n (1 + y)^n (1 + z)^n$, the sum of the co-efficients of the terms of degree 'r' is -

(A) ${}^{n^3}C_r$ (B) ${}^nC_{r^3}$ (C) ${}^{3n}C_r$ (D) $3 . {}^{2n}C_r$

12. $\binom{35}{6} + \sum\limits_{r=0}^{10}\binom{45-r}{5} = \binom{x}{y}$, then $x - y$ is equal to -

(A) 39 (B) 29 (C) 52 (D) 40

13. The value of $\sum\limits_{\substack{r=0 \\ r \le s}}^{s}\sum\limits_{s=1}^{n} {}^nC_s\, {}^sC_r$ is -

(A) $3^n - 1$ (B) $3^n + 1$ (C) 3^n (D) $3(3^n - 1)$

14. In the expansion of $\left(x^3 + 3.2^{-\log_{\sqrt{2}}\sqrt{x^3}}\right)^{11}$ -

(A) there appears a term with the power x^2

(B) there does not appear a term with the power x^2

(C) there appears a term with the power x^{-3}

(D) the ratio of the co-efficient of x^3 to that of x^{-3} is $\frac{1}{3}$

15. The sum of the series $(1 + 1).1! + (2 + 1).2! + (3 + 1).3! + ... + (n + 1).n!$ is -

(A) $(n+1) . (n + 2)!$ (B) $n . (n + 1)!$ (C) $(n+1) .(n + 1)!$ (D) none of these

16. The binomial expansion of $\left(x^k + \frac{1}{x^{2k}}\right)^{3n}$, $n \in N$ contains a term independent of x -

(A) only if k is an integer

(B) only if k is a natural number

(C) only if k is rational

(D) for any real k

17. Let $n \in N$. If $(1 + x)^n = a_0 + a_1x + a_2x^2 + ...+ a_nx^n$ and $a_{n-3}, a_{n-2}, a_{n-1}$ are in AP, then -

(A) a_1, a_2, a_3 are in AP

(B) a_1, a_2, a_3 are in HP

(C) $n = 7$

(D) $n = 14$

18. Set of values of r for which, ${}^{18}C_{r-2} + 2 . {}^{18}C_{r-1} + {}^{18}C_r \geq {}^{20}C_{13}$ contains -

(A) 4 elements (B) 5 elements (C) 7 elements (D) 10 elements

BRAIN TEASERS — ANSWER KEY — EXERCISE-2

Que.	1	2	3	4	5	6	7	8	9	10
Ans.	B	A	D	A,B	B	B	B	C	B,C	B,C,D
Que.	11	12	13	14	15	16	17	18		
Ans.	C	D	A	B,C,D	B	D	A,C	C		

EXERCISE - 03

MISCELLANEOUS TYPE QUESTIONS

FILL IN THE BLANKS

1. The greatest binomial coefficient in the expansion of $(a + b)^n$ is ________ given that the sum of all the coefficients is equal to 4096.

2. The number 7^{1995} when divided by 100 leaves the remainder ________.

3. The term independent of x in the expansion of $\left[x^2 + \frac{1}{x}\right]^{15}$ is ________ .

4. If $(1 + x + x + ... + x^p)^n = a_0 + a_1x + a_2x + ... + a_{np}\, x^{np}$ then $a_1 + 2\,a_2 + 3a_3 + ... + npa_{np}$ = ________ .

5. If $(1 + x)\,(1 + x + x^2)\,(1 + x + x^2 + x^3)\ ......\ (1 + x + x^2 + x^3 + ... + x^n) \equiv a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_mx^m$

then $\sum\limits_{r=0}^{m} a_r$ has the value equal to ________ .

6. If the 6^{th} term in the expansion of the binomial $\left[\frac{1}{x^{8/3}} + x^2 \log_{10} x\right]^8$ is 5600, then x = ________ .

7. $(1 + x)\,(1 + x + x^2)\,(1 + x + x^2 + x^3)\ ......\ (1 + x + x^2 + ... + x^{100})$ when written in the ascending power of x then the highest exponent of x is ________ .

MATCH THE COLUMN

Following question contains statements given in two columns, which have to be matched. The statements in **Column-I** are labelled as A, B, C and D while the statements in **Column-II** are labelled as p, q, r and s. Any given statement in **Column-I** can have correct matching with **ONE** statement in **Column-II**.

1.

	Column-I		Column-II
(A)	$(2n + 1)\,(2n + 3)\,(2n + 5)\ ....\ (4n - 1)$ is equal to	(p)	$\frac{(n+1)^n}{n\,!}$
(B)	$\frac{C_1}{C_0} + \frac{2\,.\,C_2}{C_1} + \frac{3\,.\,C_3}{C_2} + ... + \frac{n\,.\,C_n}{C_{n-1}}$ is equal to here C_r stand for ${}^nC_r$.	(q)	$n\,.\,2^n\,.\,(2^n - 1)$
(C)	If $(C_0 + C_1)\,(C_1 + C_2)\,(C_2 + C_3)\ ......\ (C_{n-1} + C_n)$ $= m\,.\,C_1C_2C_3\ ....\ C_{n-1}$, then m is equal to	(r)	$\frac{(4n)\,!\ n\,!}{2^n\,.\,(2n)\,!\,(2n)\,!}$
(D)	If C_r are the binomial co–efficients in the expansion of $(1 + x)^n$, the value of $\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{n}(i + j)\,C_i\,C_j$ is	(s)	$\frac{n\,(n+1)}{2}$

ASSERTION & REASON

These questions contains, Statement I (assertion) and Statement II (reason).

(A) Statement-I is true, Statement-II is true ; Statement-II is correct explanation for Statement-I.

(B) Statement-I is true, Statement-II is true ; Statement-II is NOT a correct explanation for statement-I

(C) Statement-I is true, Statement-II is false

(D) Statement-I is false, Statement-II is true

1. **Statement-I** : Coefficient of $ab^8c^3d^2$ in the expansion of $(a + b + c + d)^{14}$ is 180180

Because

Statement-II : General term in the expansion of $(a_1 + a_2 + a_3 + ... + a_m)^n$

$= \sum \frac{n!}{n_1!\,n_2!\,n_3!....n_m!} a_1^{n_1} a_2^{n_2} ... a_m^{n_m}$, where $n_1 + n_2 + n_3 + ... + n_m = n$.

(A) A (B) B (C) C (D) D

2. Statement-I : If $q = \frac{1}{3}$ and $p + q = 1$, then $\sum_{r=0}^{15} r\ {}^{15}C_r p^r q^{15-r} = 15 \times \frac{1}{3} = 5$

Because

Statement-II : If $p + q = 1$, $0 < p < 1$, then $\sum_{r=0}^{n} r\ {}^{n}C_r p^r q^{n-r} = np$

(A) A (B) B (C) C (D) D

3. Statement-I : The greatest value of ${}^{40}C_0 \cdot {}^{60}C_r + {}^{40}C_1 \cdot {}^{60}C_{r-1} \ldots\ldots {}^{40}C_{40} \cdot {}^{60}C_{r-40}$ is ${}^{100}C_{50}$

Because

Statement-II : The greatest value of ${}^{2n}C_r$, (where r is constant) occurs at r = n.

(A) A (B) B (C) C (D) D

4. Statement-I : If $x = {}^{n}C_{n-1} + {}^{n+1}C_{n-1} + {}^{n+2}C_{n-1} + \ldots\ldots + {}^{2n}C_{n-1}$, then $\frac{x+1}{2n+1}$ is integer.

Because

Statement-II : ${}^{n}C_r + {}^{n}C_{r-1} = {}^{n+1}C_r$ and ${}^{n}C_r$ is divisible by n if n and r are co-prime.

(A) A (B) B (C) C (D) D

COMPREHENSION BASED QUESTIONS

Comprehension # 1

If n is positive integer and if $(1 + 4x + 4x^2)^n = \sum_{r=0}^{2n} a_r x^r$, where a_i's are (i = 0, 1, 2, 3, ... , 2n) real numbers.

On the basis of above information, answer the following questions :

1. The value of $2\sum_{r=0}^{n} a_{2r}$ is -

(A) $9^n - 1$ (B) $9^n + 1$ (C) $9^n - 2$ (D) $9^n + 2$

2. The value of $2\sum_{r=1}^{n} a_{2r-1}$ is -

(A) $9^n - 1$ (B) $9^n + 1$ (C) $9^n - 2$ (D) $9^n + 2$

3. The value of a_{2n-1} is -

(A) 2^{2n} (B) $(n - 1).2^{2n}$ (C) $n.2^{2n}$ (D) $(n + 1).2^{2n}$

4. The value of a_2 is -

(A) $8n$ (B) $8n^2 - 4$ (C) $8n^2 - 4n$ (D) $8n - 4$

MISCELLANEOUS TYPE QUESTION — **ANSWER KEY** — EXERCISE-3

- ***Fill in the Blanks***

1. ${}^{12}C_6$ 2. 43 3. 3003 4. $\frac{np}{2}(p+1)^n$ 5. (n + 1)! 6. x = 10 7. 5050

- ***Match the Column***

1. (A)→(r), (B)→(s), (C)→(p), (D)→(q)

- ***Assertion & Reason***

1. C 2. D 3. C 4. A

- ***Comprehension Based Questions***

Comprehension # 1 : 1. B 2. A 3. C 4. C

EXERCISE - 04 [A]

CONCEPTUAL SUBJECTIVE EXERCISE

1. If the coefficients of $(2r + 4)^{th}$, $(r - 2)^{th}$ terms in the expansion of $(1 + x)^{18}$ are equal, find r.

2. If the coefficients of the r^{th}, $(r+1)^{th}$ & $(r + 2)^{th}$ terms in the expansion of $(1 + x)^{14}$ are in AP, find r.

3. Find the term independent of x in the expansion of : (a) $\left[\sqrt{\frac{x}{3}}+\frac{\sqrt{3}}{2x^2}\right]^{10}$ (b) $\left[\frac{1}{2}x^{1/3}+x^{-1/5}\right]^8$

4. Prove that : ${}^{n-1}C_r + {}^{n-2}C_r + {}^{n-3}C_r + ... + {}^{r}C_r = {}^{n}C_{r+1}$.

5. If ${}^{40}C_1 \cdot x(1-x)^{39} + 2 \cdot {}^{40}C_2\, x^2(1-x)^{38} + 3\, {}^{40}C_3\, x^3(1-x)^{37} + ... + 40 \cdot {}^{40}C_{40}\, x^{40} = ax + b$, then find a & b.

6. If ${}^{n+1}C_2 + 2\,({}^{2}C_2 + {}^{3}C_2 + {}^{4}C_2 + ... + {}^{n}C_2) = 1^2 + 2^2 + 3^2 + ... + 100^2$, then find n.

7. Which is larger : $(99^{50} + 100^{50})$ or $(101)^{50}$.

8. Show that ${}^{2n-2}C_{n-2} + 2 \cdot {}^{2n-2}C_{n-1} + {}^{2n-2}C_{n} > \frac{4n}{n+1}$, $n \in N$, $n > 2$

9. Find the coefficient of x^4 in the expansion of :

(a) $(1 + x + x^2 + x^3)^{11}$ (b) $(2 - x + 3x^2)^6$

10. Find numerically the greatest term in the expansion of :

(a) $(2 + 3x)^9$ when $x = \frac{3}{2}$ (b) $(3 - 5x)^{15}$ when $x = \frac{1}{5}$

11. Prove that the ratio of the coefficient of x^{10} in $(1 - x^2)^{10}$ & the term independent of x in $\left(x-\frac{2}{x}\right)^{10}$ is 1 : 32.

12. Find the term independent of x in the expansion of $(1 + x + 2x^3)\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9$.

13. Prove that $\sum_{k=0}^{n} {}^{n}C_k \sin Kx.\cos(n - K)x = 2^{n-1}\sin nx$.

14. Find the coefficient of :

(a) x^6 in the expansion of $(ax^2 + bx + c)^9$. (b) $x^2y^3z^4$ in the expansion of $(ax - by + cz)^9$.

(c) $a^2b^3c^4d$ in the expansion of $(a - b - c + d)^{10}$.

15. (a) $\sum_{r=0}^{20}\binom{20}{r}\binom{30}{25-r} = {}^{x}C_y$, then find x, y. (b) Prove that : $\sum_{r=0}^{25}\binom{30}{r}\binom{70}{25-r} = {}^{100}C_{25}$

16. Prove that : $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$

17. Prove that : $\sum_{r=0}^{25}(-1)^r\binom{30}{r}\binom{30}{25-r} = 0$

18. Prove that : $\sum_{r=0}^{n-2}\binom{n-1}{r}\binom{n}{r+2} = \binom{2n-1}{n-2}$

Prove the following (here $C_r = {}^nC_r$) (Q. 19 to 26) :

19. $C_0C_1 + C_1C_2 + C_2C_3 + \ldots\ldots + C_{n-1}C_n = \dfrac{(2n)!}{(n+1)!(n-1)!}$

20. $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots\ldots + C_{n-r}C_n = \dfrac{2n!}{(n-r)!(n+r)!}$

21. $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = \dfrac{(2n)!}{n!n!}$

22. $C_0 - C_1 + C_2 - C_3 + \ldots\ldots + (-1)^r.C_r = \dfrac{(-1)^r(n-1)!}{r!.(n-r-1)!}$

23. $C_1 + 2C_2 + 3C_3 + \ldots\ldots + n.\ C_n = n.\ 2^{n-1}$

24. $C_0 + 2C_1 + 3C_2 + \ldots\ldots + (n+1)\ C_n = (n+2)\ 2^{n-1}$

25. $C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \ldots\ldots + \dfrac{C_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$

26. $\dfrac{C_1}{C_0} + \dfrac{2C_2}{C_1} + \dfrac{3C_3}{C_2} + \ldots\ldots + \dfrac{n.C_n}{C_{n-1}} = \dfrac{n(n+1)}{2}$

27. Prove the identity $\dfrac{1}{{}^{2n+1}C_r} + \dfrac{1}{{}^{2n+1}C_{r+1}} = \dfrac{2n+2}{2n+1}\dfrac{1}{{}^{2n}C_r}$.

28. If $(1+x)^{15} = C_0 + C_1.\ x + C_2.\ x^2 + \ldots + C_{15}.\ x^{15}$ and $C_2 + 2C_3 + 3C_4 + \ldots + 14C_{15} = a2^b + c$, then find $a+b+c$.

29. Evaluate : $2^{15}\binom{30}{0}\binom{30}{15} - 2^{14}\binom{30}{1}\binom{29}{14} + 2^{13}\binom{30}{2}\binom{28}{13} \ldots\ldots - \binom{30}{15}\binom{15}{0}$

ANSWER KEY

CONCEPTUAL SUBJECTIVE EXERCISE — EXERCISE-4(A)

1. $r = 6$ **2.** $r = 5$ or 9 **3.** (a) $T_3 = \dfrac{5}{12}$ (b) $T_6 = 7$ **5.** $a = 40$, $b = 0$ **6.** 100

7. 101^{50} **9.** (a) 990 (b) 3660 **10.** (a) $T_7 = \dfrac{7.3^{13}}{2}$ (b) 455×3^{12} **12.** $\dfrac{17}{54}$

14. (a) $84b^6c^3 + 630ab^4c^4 + 756a^2b^2c^5 + 84a^3c^6$; (b) $-1260.a^2b^3c^4$; (c) -12600

15 (a) $x = 50$, $y = 25$ **28.** 28 **29.** $\binom{30}{15}$

EXERCISE - 04 [B]

BRAIN STORMING SUBJECTIVE EXERCISE

1. If $\sum_{r=0}^{2n} a_r (x-2)^r = \sum_{r=0}^{2n} b_r (x-3)^r$ & $a_k = 1$ for all $k \ge n$, then show that $b_n = {}^{2n+1}C_{n+1}$

2. Prove the following :

(a) $C_0^{\,2} - C_1^{\,2} + C_2^{\,2} - C_3^{\,2} + \ldots\ldots + (-1)^n C_n^{\,2} = \begin{cases} 0 & \text{if n is odd} \\ (-1)^{n/2}\ {}^nC_{n/2} & \text{if n is even} \end{cases}$

(b) $1.C_0^{\,2} + 3.C_1^{\,2} + 5.C_2^{\,2} + \ldots\ldots + (2n+1)C_n^{\,2} = \dfrac{(n+1)(2n)!}{n!n!}$

3. (a) Find the index n of the binomial $\left(\dfrac{x}{5}+\dfrac{2}{5}\right)^n$ if the 9^{th} term of the expansion has numerically the greatest coefficient ($n \in N$).

(b) For which positive values of x is the fourth term in the expansion of $(5+3x)^{10}$ is the greatest.

4. If $a_0, a_1, a_2, \ldots\ldots$ be the coefficients in the expansion of $(1+x+x^2)^n$ in ascending powers of x, then prove that :

(a) $a_0a_1 - a_1a_2 + a_2a_3 - \ldots. = 0$

(b) $a_0a_2 - a_1a_3 + a_2a_4 - \ldots\ldots + a_{2n-2}\, a_{2n} = a_{n+1}$ or a_{n-1}

(c) $E_1 = E_2 = E_3 = 3^{n-1}$; where $E_1 = a_0 + a_3 + a_6 + \ldots\ldots$; $E_2 = a_1 + a_4 + a_7 + \ldots\ldots$ & $E_3 = a_2 + a_5 + a_8 + \ldots\ldots$

5. Prove that : $1^2 . C_0 + 2^2 C_1 + 3^2 . C_2 + 4^2 . C_3 + \ldots\ldots + (n+1)^2 . C_n = 2^{n-2}(n+1)(n+4)$.

6. If $(1+x)^n = \sum_{r=0}^{n} C_r . x^r$ then prove that ; $\dfrac{2^2.C_0}{1.2} + \dfrac{2^3.C_1}{2.3} + \dfrac{2^4.C_2}{3.4} + \ldots\ldots + \dfrac{2^{n+2}.C_n}{(n+1)(n+2)} = \dfrac{3^{n+2}-2n-5}{(n+1)(n+2)}$

7. Prove that : $\sum_{i=0}^{r}\binom{n+i}{k} = \binom{n+r+1}{k+1} - \binom{n}{k+1}$

8. Prove that : $\sum_{j=0}^{\infty}\binom{p}{j}\binom{q}{n+j} = \binom{p+q}{p+n}$, $p, q \in N$; p, q are constants.

9. Prove that : $\sum_{r=1}^{n}\binom{n-1}{n-r}\binom{n}{r} = \binom{2n-1}{n-1}$

10. Prove that : $\dfrac{C_0}{2} + \dfrac{C_1}{3} + \dfrac{C_2}{4} + \dfrac{C_3}{5} + \ldots\ldots + \dfrac{C_n}{n+2} = \dfrac{1+n.\ 2^{n+1}}{(n+1)(n+2)}$

11. Prove that : $\dfrac{1}{2}\,{}^nC_1 - \dfrac{2}{3}\,{}^nC_2 + \dfrac{3}{4}\,{}^nC_3 - \dfrac{4}{5}\,{}^nC_4 + \ldots.. + \dfrac{(-1)^{n+1}n}{n+1}.{}^nC_n = \dfrac{1}{n+1}$

12. Prove that : $({}^{2n}C_1)^2 + 2.({}^{2n}C_2)^2 + 3.({}^{2n}C_3)^2 + \ldots.. + 2n.({}^{2n}C_{2n})^2 = \dfrac{(4n-1)!}{[(2n-1)!]^2}$

BRAIN STORMING SUBJECTIVE EXERCISE	ANSWER KEY	EXERCISE-4(B)
3. (a) $n = 12$ (b) $\dfrac{5}{8} < x < \dfrac{20}{21}$		

EXERCISE - 05 [A]

JEE-[MAIN] : PREVIOUS YEAR QUESTIONS

1. The sum of the coefficients in the expansion of $(x + y)^n$ is 4096. The greatest coefficient in the expansion is- [AIEEE 2002]

(1) 1024 (2) 924 (3) 824 (4) 724

2. If for positive integers r > 1, n > 2 the coefficients of the $(3r)^{th}$ and $(r+2)^{th}$ powers of x in the expansion of $(1+x)^{2n}$ are equal, then- [AIEEE 2002]

(1) n = 2r (2) n = 3r (3) n = 2r + 1 (4) n = 2r − 1

3. If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots + \ldots C_nx^n$, then $\frac{C_1}{C_0} + \frac{2C_2}{C_1} + \frac{3C_3}{C_2} + \ldots + \frac{nC_n}{C_{n-1}} =$ [AIEEE-2002]

(1) $\frac{n(n-1)}{2}$ (2) $\frac{n(n+2)}{2}$ (3) $\frac{n(n+1)}{2}$ (4) $\frac{(n-1)(n-2)}{2}$

4. The number of integral terms in the expansion of $\left(\sqrt{3}+\sqrt[8]{5}\right)^{256}$ is- [AIEEE 2003]

(1) 32 (2) 33 (3) 34 (4) 35

5. The coefficient of the middle term in the binomial expansion in powers of x of $(1 + \alpha x)^4$ and of $(1 - \alpha x)^6$ is the same if α equals- [AIEEE 2004]

(1) $-\frac{5}{3}$ (2) $\frac{10}{3}$ (3) $\frac{-3}{10}$ (4) $\frac{3}{5}$

6. The coefficient of x^n in expansion of $(1 + x)(1 - x)^n$ is- [AIEEE 2004]

(1) (n − 1) (2) $(-1)^n (1-n)$ (3) $(-1)^{n-1} (n-1)^2$ (4) $(-1)^{n-1}n$

7. If the coefficients of r^{th}, $(r + 1)^{th}$ and $(r + 2)^{th}$ terms in the binomial expansion $(1 + y)^m$ are in A.P., then m and r satisfy the equation- [AIEEE 2005]

(1) $m^2 - m(4r - 1) + 4r^2 + 2 = 0$ (2) $m^2 - m(4r + 1) + 4r^2 - 2 = 0$

(3) $m^2 - m(4r + 1) + 4r^2 + 2 = 0$ (4) $m^2 - m(4r - 1) + 4r^2 - 2 = 0$

8. If the coefficient of x^7 in $\left[ax^2 + \left(\frac{1}{bx}\right)\right]^{11}$ equals the coefficient of x^{-7} in $\left[ax - \left(\frac{1}{bx^2}\right)\right]^{11}$, then a and b satisfy the relation- [AIEEE 2005]

(1) ab = 1 (2) $\frac{a}{b} = 1$ (3) a + b = 1 (4) a − b = 1

9. For natural numbers m, n if $(1 - y)^m (1 + y)^n = 1 + a_1y + a_2y^2 + \ldots\ldots$, and $a_1 = a_2 = 10$, then (m, n) is- [AIEEE 2006]

(1) (45, 35) (2) (35, 45) (3) (20, 45) (4) (35, 20)

10. The sum of the series ${}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + \ldots\ldots - \ldots\ldots + {}^{20}C_{10}$ is - [AIEEE 2007]

(1) $\frac{1}{2}\,{}^{20}C_{10}$ (2) 0 (3) $-{}^{20}C_{10}$ (4) ${}^{20}C_{10}$

11. In the binomial expansion of $(a - b)^n$, $n \geq 5$, the sum of 5^{th} and 6^{th} terms is zero, then $\frac{a}{b}$ equals [AIEEE 2007]

(1) $\frac{6}{n-5}$ (2) $\frac{n-5}{6}$ (3) $\frac{n-4}{5}$ (4) $\frac{5}{n-4}$

12. **Statement –1** : $\sum_{r=0}^{n}(r+1)\,{}^{n}C_r = (n+2)2^{n-1}$

Statement–2 : $\sum_{r=0}^{n}(r+1)\,{}^{n}C_r x^r = (1 + x)^n + nx\ (1+x)^{n-1}$ **[AIEEE 2008]**

(1) Statement –1 is false, Statement –2 is true

(2) Statement–1 is true, Statement–2 is true; Statement–2 is a correct explanation for Statement–1

(3) Statement–1 is true, Statement–2 is true; Statement–2 is not a correct explanation for Statement–1

(4) Statement–1 is true, Statement–2 is false

13. The remainder left out when $8^{2n} - (62)^{2n+1}$ is divided by 9 is :- **[AIEEE 2009]**

(1) 7 (2) 8 (3) 0 (4) 2

14. Let $S_1 = \sum_{j=1}^{10} j(j-1)\,{}^{10}C_j$, $S_2 = \sum_{j=1}^{10} j\,{}^{10}C_j$ and $S_3 = \sum_{j=1}^{10} j^2\,{}^{10}C_j$. **[AIEEE-2010]**

Statement–1 : $S_3 = 55 \quad 2^9$.

Statement–2 : $S_1 = 90 \quad 2^8$ and $S_2 = 10 \quad 2^8$.

(1) Statement–1 is true, Statement–2 is true; Statement–2 is a correct explanation for Statement–1.

(2) Statement–1 is true, Statement–2 is true; Statement–2 is not a correct explanation for statement–1.

(3) Statement–1 is true, Statement–2 is false.

(4) Statement–1 is false, Statement–2 is true.

15. The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$ is :- **[AIEEE 2011]**

(1) –144 (2) 132 (3) 144 (4) – 132

16. If n is a positive integer, then $\left(\sqrt{3}+1\right)^{2n} - \left(\sqrt{3}-1\right)^{2n}$ is : **[AIEEE 2012]**

(1) a rational number other than positive integers (2) an irrational number

(3) an odd positive integer (4) an even positive integer

17. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1} - \frac{x-1}{x-x^{1/2}}\right)^{10}$ is : **[JEE (Main)-2013]**

(1) 4 (2) 120 (3) 210 (4) 310

PREVIOUS YEARS QUESTIONS **ANSWER KEY** EXERCISE-5 [A]

Que.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ans	2	3	3	2	3	2	2	1	2	1	3	2	4	3	1
Que.	16	17													
Ans	2	3													

EXERCISE - 05 [B] JEE-[ADVANCED] : PREVIOUS YEAR QUESTIONS

1. (a) For $2 \le r \le n$, $\binom{n}{r} + 2\binom{n}{r-1} + \binom{n}{r-2} =$ [JEE 2000, (Screening), 1+1M]

(A) $\binom{n+1}{r-1}$ (B) $2\binom{n+1}{r+1}$ (C) $2\binom{n+2}{r}$ (D) $\binom{n+2}{r}$

(b) In the binomial expansion of $(a - b)^n$, $n \ge 5$, the sum of the 5^{th} and 6^{th} terms is zero, Then $\frac{a}{b}$ equals -

(A) $\frac{n-5}{6}$ (B) $\frac{n-4}{5}$ (C) $\frac{5}{n-4}$ (D) $\frac{6}{n-5}$

2. For any positive integers m, n (with $n \ge m$) , let $\binom{n}{m} = {}^nC_m$. Prove that :

$$\binom{n}{m} + \binom{n-1}{m} + \binom{n-2}{m} + \ldots\ldots.. + \binom{m}{m} = \binom{n+1}{m+1}$$

Hence or otherwise prove that, [JEE 2000 (Mains), 6M]

$$\binom{n}{m} + 2\binom{n-1}{m} + 3\binom{n-2}{m} + \ldots\ldots.. + (n-m+1)\binom{m}{m} = \binom{n+2}{m+2}.$$

3. The sum $\sum_{i=0}^{m}\binom{10}{i}\binom{20}{m-i}$ (where $\binom{p}{q} = 0$) if $p < q$ is maximum when m is - [JEE 02(Screening), 3M]

(A) 5 (B) 10 (C) 15 (D) 20

4. (a) Coefficient of t^{24} in the expansion of $(1 + t^2)^{12} (1 + t^{12}) (1 + t^{24})$ is - [JEE 03, Screening , 3M out of 60]

(A) ${}^{12}C_6 + 2$ (B) ${}^{12}C_6 + 1$ (C) ${}^{12}C_6$ (D) none

(b) If n and k are positive integers, show that [JEE 03, Mains 2M out of 60]

$$2^k\binom{n}{0}\binom{n}{k} - 2^{k-1}\binom{n}{1}\binom{n-1}{k-1} + 2^{k-2}\binom{n}{2}\binom{n-2}{k-2} \ldots..(-1)^k\binom{n}{k}\binom{n-k}{0} = \binom{n}{k}$$

5. If $n, r \in N$ and ${}^{n-1}C_r = (k^2 - 3)\,({}^nC_{r+1})$, then k lies in the interval -

[JEE 04, Screening, 3M out of 84]

(A) $\left[-\sqrt{3}, \sqrt{3}\right]$ (B) $(2, \infty)$ (C) $\left[-\sqrt{3}, \infty\right]$ (D) $\left(\sqrt{3}, 2\right]$

6. The value of $\binom{30}{0}\binom{30}{10} - \binom{30}{1}\binom{30}{11} + \binom{30}{2}\binom{30}{12} \ldots\ldots\ldots + \binom{30}{20}\binom{30}{30}$, is where $\binom{n}{r} = {}^nC_r$

[JEE 05, Screening, 3M out of 84]

(A) ${}^{30}C_{10}$ (B) ${}^{60}C_{20}$ (C) ${}^{31}C_{11}$ or ${}^{31}C_{10}$ (D) ${}^{30}C_{11}$

7. For r = 0, 1,....,10, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1+x)^{10}$, $(1 + x)^{20}$ and $(1 + x)^{30}$. Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to - [JEE 10, 5M, –2M]

(A) $B_{10} - C_{10}$ (B) $A_{10}\left(B_{10}^2 - C_{10}A_{10}\right)$ (C) 0 (D) $C_{10} - B_{10}$

8. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n =

[JEE-Advanced 2013, 4, (–1)]

PREVIOUS YEARS QUESTIONS		ANSWER KEY			EXERCISE-5 [B]	
1. (a) D ; (b) B	3. C	4. (a) A	5. D	6. A	7. D	8. 6